May-Wigner Instabilty Scenario :

"Will a Large Complex System be Stable?"

This question was posed by **Robert May** (*NATURE* **238**, 413 (1972)) who introduced a toy **linear** model for (in)stability of a large system of many interacting species:

$$\dot{x} = \quad x + Bx; \qquad > 0; \; x \in \mathbb{R}^N$$

Without interactions the part $\dot{x} = \quad x$ describes a **simple exponential** relaxation of N uncoupled degrees of freedom x_i with the same rate $\; > 0$ towards the **stable equilibrium** $x = 0$. A complicated interaction between dynamics of different degrees of freedom is mimicked by a general **real asymmetric** $N \quad N$ **random matrix** B with mean zero and prescribed variance $\; ^2$ of all entries. As a typical eigenvalue of B with the largest real part grows as $\sqrt{N}$ the equilibrium $x = 0$ becomes **unstable** as long as $\; < \sqrt{N}$.

This scenario is known in the literature as the "**May-Wigner** instability" and despite its oversimplifying and schematic nature attracted very considerable attention in mathematical ecology and complex systems theory over the years.

Counting equilibria via Kac-Rice formulae:

A standard analysis of autonomous ODE's starts with finding **equilibrium points** and classifying them by **stability** properties.

We would like to know the **total number** $N_{tot}(D)$ of all possible **equilibria** of our system of nonlinear ODEs, i.e. the number of simultaneous solutions of N equations

$$x_i + f_i(x_1; : : : ; x_N$$

Mean number of equilibria and the Elliptic Ensemble:

Using Kac-Rice approach we are able to count the **mean** total number $E\{N_{tot}\}$ of all possible **equilibria** in the system of nonlinear ODEs under consideration. This turns out to be given by (**YVF** & **Khoruzhenko**, *in progress*):

$$E\{N_{tot}\} = \frac{1}{m^N N^{(N-1)/2}} \int_{1}^{1} \left\langle \det \left(m + t\sqrt{\ } \right) \sqrt{N} \quad X \right\rangle_X \frac{e^{\frac{Nt^2}{2}} dt}{\sqrt{2}}$$

where $m = \quad = {}_c$ with some characteristic scale ${}_c$, and the random **real asymmetric** matrix X being taken from the **Gaussian Elliptic Ensemble**:

$$P(X) = C_N(\ )e^{\frac{1}{2(1 \quad ^2)}[\mathrm{Tr}XX^T \quad \mathrm{Tr}X^2]}, \qquad 2\,[0;1]$$

The parameter depends on the ratio of variances of **gradient** and **solenoidal** components of the field such that the **real Ginibre ensemble** with $= 0$ corresponds to **purely solenoidal**, and GOE with $= 1$ to **purely gradient** flow.

Let us denote ${}^{(r)}_{N}(\ )$ the mean density of **real** eigenvalues of $N \quad N$ matrices X for the elliptic ensemble at . Then it turns out that (cf. **Edelman, Kostlan, Schub** '94.)

$$\langle |\det(\quad X)| \rangle_X = 2\sqrt{1 + \quad} \frac{(N \quad 1)!}{(N \quad 2)!!} \, {}^{(r)}_{N+1}(\ )\, e^{\frac{\ ^2}{\quad}}$$

A Nonlinear Analogue of May-Wigner Instability as Topology Detrivialization:

The mean density $^{(r)}_{N}(\ )$ of real eigenvalues for the elliptic ensemble was computed explicitly by **Forrester** & **Nagao** ’08 in terms of Hermite polynomials, and its large-N asymptotic behaviour was studied as well.

Asymptotic analysis of the counting problem for $N \quad 1$ reveals then a **topology detrivialization** transition, with the total number of equilibria **abruptly** changing from **a single equilibrium** for $\ > \ _{c} = \sqrt{N(F''(0) + \ ''(0))}$ to **exponentially many** equilibria as long as $\ < \ _{c}$:

$$\mathrm{E}\{N_{tot}\} \quad \sqrt{\frac{2(1+\ )}{1\ }}\, e^{N\ _{tot}(m)}; \qquad _{tot}(m) = \frac{m^2\ \ 1}{2} \quad \ln m > 0 \quad \text{for } m = \frac{\ }{\ _{c}} < 1$$

Landscape topology (de)trivialization for gradient dynamics:

In the case of purely **gradient** dynamics $\underline{x} = \quad x \quad r\,V(x) = \quad r\,L(x)$ where :

$$L(x) = \frac{\;}{2} \sum_{i=1}^{N \;\; 1} x_i^2 + V(x_1, \dots, x_{N \;\; 1}); \qquad > 0; \quad 1 < x_i < 1$$

is the Lyapunov function (or "energy functional"). Correspondingly the equilibria points are simply **stationary** points of the Lyapunov function whereas the stable equilibria are local **minima**.

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Taking as before $V(x)$ to be **stationary isotropic** random Gaussian field with covariance structure $E\{V(x)V(y)\} = F\;\;(x$

Landscape topology (de)trivialization for gradient dynamics:

The asymptotics $F_{N \; 1}(t)$ is well known (**Tracy** & **Widom** ’94; **Borot et al** ’11). Using it for a fixed $m \neq 1$ we find for the mean number of minima:

$$E\{N_{min}\} \qquad \begin{array}{ll} 1; & m > 1 \\ e^{N \;\; st(m)}; & m < 1 \end{array}$$

Here the complexity of stable equilibria (minima) is given by

$$(m) = \quad \tfrac{1}{2}(m^2$$

Part II: Topology of Random Algebraic Varieties :

Recently, the problem of computing the expectation of topological properties of random algebraic varieties has attracted a lot of interest (see e.g. the works by **Burgisser '07, Nazarov-Sodin '09, Gayet-Welshinger '11, Sarnak '11, Lerario-Lundberg '12, Sarnak-Wigman '13**) and others. An important class of problems addresses estimates for Betti numbers of "generic" (=random) real hypersurfaces given by **zero set** of real random homogenious polynomials of degree d in $n+1$ variables restricted to the unit sphere. E.g. for $d = 60$ and $n = 2$ a typical picture is:

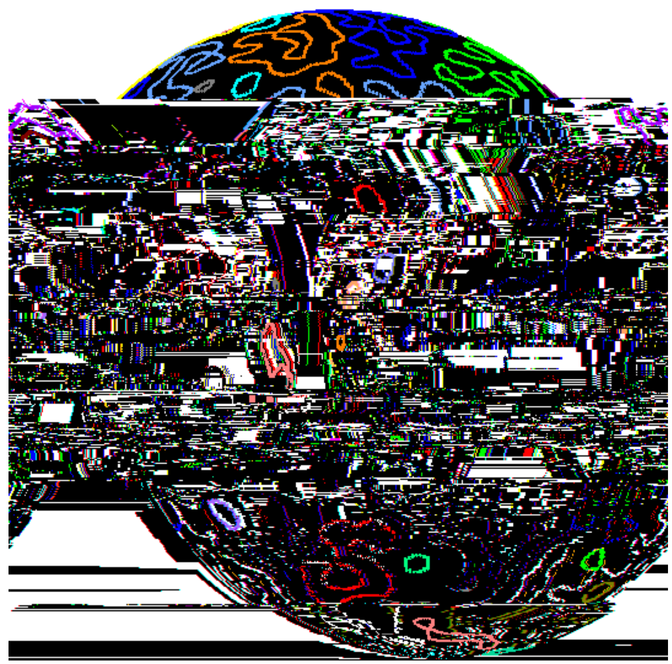

Figure 1: Zero locus of a random polynomial of degree $d = 60$ on the sphere (**M. Nastasescu**)

Upper bound on b_0 by Random Matrix Theory:

It turns out that the methods and results just exposed allow one to provide a useful **upper bound** to the **expected number of connected components** $b_0(f)$. Indeed, every component of the zero locus of the polynomials restricted to the sphere bounds a region where the function attains at least a maximum or a minimum, and consequently $\mathbb{E}\{b_0(f)\} \leq \mathbb{E}\{N_{min} + N_{max}\}$, where $N_{min/max}$ are numbers of minima/maxima on the sphere. The problem then amounts to counting minima of a random function on a sphere.

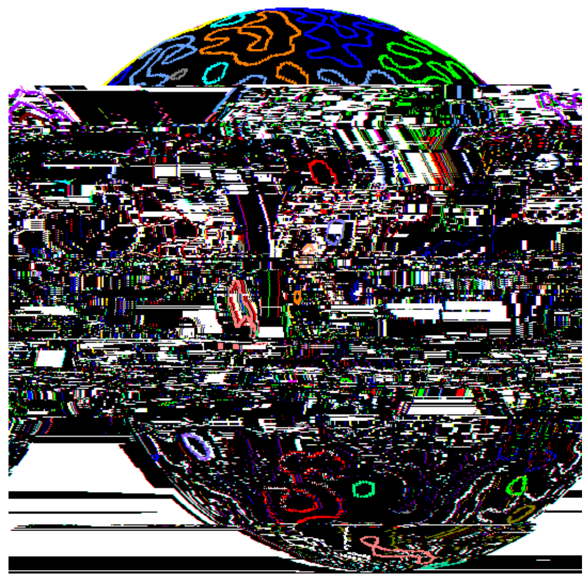

Figure 2: Zero locus of a random polynomial of degree $d = 60$ on the sphere (**M. Nastasescu**)

Counting Stationary points for Isotropic Gaussian Landscapes:

In recent years there was a steady progress in counting & classifying the **mean number** of **stationary points** of smooth isotropic Gaussian random fields $V(\mathbf{x})$ on the sphere $|\mathbf{x}| = R$ such that

$$\mathbb{E}\{V(\mathbf{x})V(\mathbf{x}')\} = F(\mathbf{x}-\mathbf{x}')$$

Using the multidimensional **Kac-Rice** integrals it was shown, in particular, that $\mathbb{E}\{N_{min}\}$ can be again directly related to the the distribution $F_N(t)$ of the maximal eigenvalue of **random GOE** matrices H such that $P(H) \propto \exp\left(-\frac{N}{4}\mathrm{Tr}H^2\right)$. Namely

$$\mathbb{E}\{N_{min}\} = 2\left(\frac{1+B}{1-B}\right)^{N/2}\sqrt{1-B}\int_{-\infty}^{\infty} e^{-NBt^2}$$

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Summary :

For the special case of **purely gradient** flows one can also find explicit expression for the number of **stable** equilibria. The latter are exponential in N but their fraction among all equilibria is negligible. The crossover expression in that case is given in terms of the