Random matrices with equi-spaced external source

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joint work with Dong Wang (University of Singapore)

Random matrices with external source

- space of **n × n** Hermitian matrices with probability measure

$$\frac{}{\mathbf{Z}_n} \text{xp}\left(-\mathbf{n}\mathrm{Tr}\left(\mathbf{V}\left(\mathbf{M} \quad - \mathbf{AM} \quad \mathbf{dM},\right.\right.\right.$$

where

- ▸ **V** is a polynomial of even degree with positive leading

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- if $\mathbf{A}$, unitary ensemble

$$\frac{}{\mathbf{Z}_n} \mathrm{xp}\left(-\mathbf{n}\mathrm{Tr}\,\mathbf{V}\left(\mathbf{M}\right.\right. \quad \mathbf{dM}.$$

- we will study the case

$$\mathbf{A} \quad \frac{}{\mathbf{n}}\mathrm{diag}\left(\;,\; \;, \ldots, \mathbf{n} - \right.$$

  - for $\mathbf{V}\left(\mathbf{x}\right. \quad \mathbf{cx}$, eigenvalues behave like $\mathbf{n}$ non-intersecting Brownian motions starting at and ending at $,\frac{1}{n},\frac{}{n},\ldots,\frac{n-1}{n}$

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- Joint probability distribution of eigenvalues in the ensemble

$$\frac{\quad}{Z_n}\ \mathrm{xp}\big(-n\,\mathrm{Tr}\,\big(V\,\big(M \quad - AM \quad dM$$

is given by

$$\frac{\quad}{Z_n}\Big|\ \mathrm{t}\big(e^{na_i\lambda_j}\ {}_{i,j=1,\ldots,n}\Big| \prod_{i<j} |\lambda_i - \lambda_j| \prod_{j=1}^{n} e^{-nV(\lambda_j)}d\lambda_j$$

- if **A** $\frac{1}{n}$(0a1

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- **A** diag$(\mathbf{a}, \ldots, \mathbf{a}, -\mathbf{a}, \ldots, -\mathbf{a}$ *(Bleher-Kuijlaars, Bleher-Delvaux-Kuijlaars, Adler-van Moerbeke)*
  - vector equilibrium problem
  - critical point: Pearcey kernel
- **A** diag$(\mathbf{a}_1, \mathbf{a}\ , \ldots, \mathbf{a}\ , \ \ , \ldots,$ with **k** fixed *(Baik-Wang, Bertola-Buckingham-Lee-Pierce, Adler-Délépine-van Moerbeke)*
  - every non-zero eigenvalue of **A** is responsable for at most one outlier-eigenvalue of **M**
- External source matrix with **n** different eigenvalues *(Eynard-Orantin)*

External source

- **A** $\frac{1}{n}\mathrm{diag}\Big($ **, ,...,n− ,n−**

Limiting mean eigenvalue density

- limiting mean distribution minimizes

$$-\iint \log|t-s|^{-1}\,d\mu(t)\,d\mu(s) \quad - \iint \log|e$$

Eigenvalue correlation kernel

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$\mathbf{A} \quad \frac{1}{n}\mathrm{diag}(\ , \ , \dots, \mathbf{n} - \ , \mathbf{n} - \quad :$

- correlation kernel for eigenvalues is given by

$$\mathbf{K}_n(\mathbf{x},\mathbf{y} \qquad \mathbf{e}^{-\frac{n}{2}V(x} \ \mathbf{e}^{-\frac{n}{2}V(} \ \mathop{}_{=0}^{n-1} \mathbf{p} \ (\mathbf{x} \ \mathbf{q} \ (\mathbf{e}$$

  - polynomials $\mathbf{p}_k$ of degree **k** and $\mathbf{q}_j$ of degree **j** are determined by the orthogonality conditions

$$\int \mathbf{p}_k(\mathbf{x} \ \mathbf{q}_j(\mathbf{e}^x \ \mathbf{e}^{-nV(x)}\mathbf{dx} \qquad \boldsymbol{\delta}_{kj}$$

  - $\mathbf{p}_k$'s are type II multiple OPs with **n** orthogonality weights $, \mathbf{e}^x, \mathbf{e}^{2x}, \dots, \mathbf{e}^{(n-1)x}$

Eigenvalue correlation kernel

Random matrices with external source

$\mathbf{A} \quad \frac{1}{n}\text{diag}(\quad , \quad , \ldots , \mathbf{n} - \quad , \mathbf{n} - \quad :$

- correlation kernel for eigenvalues is given by

$$\mathbf{K}_n(\mathbf{x}, \mathbf{y} \qquad \mathbf{e}^{-\frac{n}{2}V(x} \mathbf{e}^{-\frac{n}{2}V(} \sum_{=0}^{n-1} \mathbf{p} \ (\mathbf{x} \ \mathbf{q} \ (\mathbf{e}$$

  - polynomials $\mathbf{p}_k$ of degree $\mathbf{k}$ and $\mathbf{q}_j$ of degree $\mathbf{j}$ are determined by the orthogonality conditions

$$\int \mathbf{p}_k(\mathbf{x} \ \mathbf{q}_j(\mathbf{e}^x \ \mathbf{e}^{-nV(x)}\mathbf{dx} \qquad \boldsymbol{\delta}_{kj}$$

  - $\mathbf{q}_j$'s are related to type I multiple orthogonal polynomials

Eigenvalue correlation kernel

Interpretation of the polynomials in terms of the random matrix ensemble

$$\frac{1}{Z_n} \exp\left(-n \operatorname{Tr}\left(V(M) - AM\right)\right) dM$$

or the determinantal point process

$$\frac{1}{Z_n} \prod_{i<j} (\lambda_i - \lambda_j) \prod_{i<j} (e^{\lambda_i} - e^{\lambda_j}) \prod_{j=1}^{n} e^{-nV(\lambda_j)} d\lambda_j .$$

- **p**

Eigenvalue correlation kernel

Eigenvalue correlation kernel

- RH problem for usual OPs *(Fokas-Its-Kitaev '92)*

(a) Y is analytic in $\mathbb{C} \setminus \mathbb{R}$,

(b) $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}$,

(c) $Y(z) = (I + O(z^{-1})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$,

- Unique solution given by

$$Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \kappa_n^{-1} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{p_n(s) w(s)}{s - z} ds \\ -2\pi i \kappa_{n-1} p_{n-1}(z) & -\kappa_{n-1} \int_{\mathbb{R}} \frac{p_{n-1}(s) w(s)}{s - z} ds \end{pmatrix},$$

Eigenvalue correlation kernel

- polynomials defined by

$$\int_{\mathbb{R}} p\ (x\ q\ (e^{x}\ e^{-nV(x}\ dx \qquad \delta$$

- standard RH problem for MOPs is of size **n** - inconvenient for **n** large
- let

$$Y_1(z \qquad \kappa_n^{-1} p_n(z$$

and

$$Y\ (z \qquad \frac{-\kappa_n^{-1}}{\pi i} \int_{\mathbb{R}} \frac{p_n(s}{e\ - e^{s}} e^{-nV(s}\ ds.$$

-

RH problem for polynomials

1. **Y** $(\mathbf{Y}_1, \mathbf{Y}_2)$, where $\mathbf{Y}_1$ is analytic in $\mathbb{C} \setminus \mathbb{R}$, and $\mathbf{Y}_2$ is analytic in

RH problem for polynomials

- there is also a $\times$ matrix RH problem
  - unlike for usual orthogonal polynomials,
    $\mathrm{t}\,\mathbf{Y}(\mathbf{z}$ □
  - taking inverses is not possible
  - no advantage
- there is a dual RH problem for $\mathbf{Y}$ $(\mathbf{Y}_1, \mathbf{Y}$, where

$$\mathbf{Y}_1 \quad -\kappa_n q_n(\mathbf{e} \quad , \qquad \mathbf{Y}\ (\mathbf{z} \qquad \frac{-\kappa_n}{\pi i} \int \frac{q_n(\mathbf{e}^s}{\mathbf{z}-\mathbf{s}} \mathbf{e}^{-nV(s} \, \mathbf{ds}.$$

RH problem for polynomials

1. $\mathbf{Y}$ $(\mathbf{Y}_1, \mathbf{Y}_2$, where $\mathbf{Y}_2$

RH problem for polynomials

- Asymptotic analysis of the RH problem if the support of **μ** is one interval: Deift/Zhou steepest descent analysis
- Modifications compared to analysis for OPs
  - construction of two **g**-functions

$$g(z) = \int \log(z - y)\, d\mu(y)$$

$$g(z) = \int \log(e^{z} - e^{y})\, d\mu(y).$$

  - Crucial step: transformation of the RH problem to a non-local scalar RH problem in the complex plane

RH problem for polynomials

- Transformation to shifted RH problem of the form

1. $\mathbf{F}$ $\mathbb{C}$ $\backslash$ $\to$ $\mathbb{C}$ is analytic
2. for $\mathbf{z}$ □ , we have

$$\mathbf{F}_+(\mathbf{z} \quad \mathbf{F}_-(\mathbf{z}\ \mathbf{J}_n(\mathbf{z} \quad \mathbf{F}_\pm(\mathbf{f}(\mathbf{z}\ \ \mathbf{J}_n(\mathbf{z}\ ,$$

   with $\mathbf{f}$ $\to$,

3.

Outlook

- Universality
  - sine kernel
  - Airy kernel
- multi-cut case
- large **n** behavior in more general point processes of the form

$$\frac{1}{Z_n} \prod_{i<j} (\lambda_i - \lambda_j) \prod_{i<j} (f(\lambda_i) - f(\lambda_j)) \prod_{j=1}^{n} e^{-nV(\lambda_j)} d\lambda_j.$$