

The Chiral Gaussian Two-Matrix Ensemble of Real Asymmetric Matrices

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Definition of the New Ensemble

2 Solve for eigenvalue correlation functions of the matrix

$$D = \begin{array}{c|c} \begin{array}{c} 0 \\ \text{B} \end{array} & \begin{array}{c} 0 \\ P^T \end{array} \\ \hline \begin{array}{c} P \\ j \end{array} & \begin{array}{c} 1 \\ Q^T \\ 0 \end{array} \end{array} \begin{array}{c} 1 \\ Q \\ A \end{array}$$

2 P and Q are **real-valued** matrices (i.e. $\tau = 1$) of size $N \times (N + \rho)$, with elements independently-distributed Gaussian $\gg N[0;1]$.

2 $\tau \in [0;1]$ is the **non-Hermiticity parameter**. $\tau = 0$, chGOE [3].

2 Model has ρ eigenvalues which are **precisely zero**.

2 Other eigenvalues come in

- { **pairs** (S^{α_j}), either both real or both imaginary, or
- { complex-valued **quadruplets** ($S^{\alpha_j}; S^{\alpha_j^*}$).

2 Model finds applications in 2-colour **Quantum Chromodynamics** (QCD) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100]

The Joint PDF (P_N) for Finite N

We solve for the eigenvalues of the $N \times N$ **Wishart-type** matrix:

$$W = (P + iQ)(P^T - iQ^T)$$

whose entries are **non-Gaussian** and **correlated**. The Dirac eigenvalues are the square roots of Wishart eigenvalues ($z = x + iy = \alpha^2$).

$N = 1$

One real (Wishart) eigenvalue

$$P_1(x) \gg |x|$$

The Algebraic Structure of $\beta = 1$ Ensembles

P_N has the same form as the **non-chiral Ginibre** crossover ensemble (see [11, 12] and references therein). Therefore, we can immediately write the partition function Z and correlation functions R_n as follows:

$$Z \gg \text{Pf} \int d^2 z_1 \int d^2 z_2 F(z_1; z_2) z_1^{j-1} z_2^{i-1}$$

Calculation of the Kernel using Grassmann Integrations

We can show that the kernel K_N is related to the expectation of the product of two **characteristic polynomials**:

$$K_N(u; v) \approx (u; v) \frac{\det(\rho_{\bar{u}j} D) \det(\rho_{\bar{v}j} D) i_{N_j 2}}{(uv)^{\circ=2}}$$

To evaluate this expectation:

2 Replace the determinants with Grassmann integrals

$$\det M = \int d^{\prime} e^{i \sum_i M_{ij} \hat{j}_i}$$

2

Eigenvalue Densities at Finite N

The Strong Large- N Limit

For physical applications, large- N limits are usually required. For the **strong** limit, we keep ρ fixed. No scaling of eigenvalues is required if we are interested in behaviour close to the origin. Using the Hardy-Hille formula (for weighted sums of Laguerre polynomials), we find:

$$\frac{1}{2}C(z) \gg \int_0^z \frac{dt}{t} e^{i \frac{1}{4t} (z^2 + z^2 t^2)} K_{\nu=2} (2 \sqrt{t} z) \operatorname{erfc} \left(2 \sqrt{t} z \right)$$

The Weak Large- N Limit

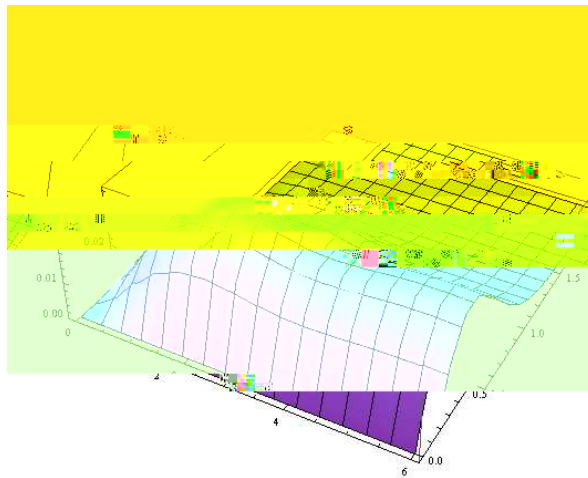
Here, we scale ρ with $\rho \frac{1}{N}$ (i.e. $\rho \neq 0$). It is also necessary to scale the Dirac eigenvalues by $\rho \frac{1}{N}$ if we are to reach an interesting limit:

$$\begin{aligned} \rho &= \frac{\rho^{\text{Dirac}}}{2N} && \text{for fixed } \rho^{\text{Dirac}}; \\ z &= \frac{z^{\text{Dirac}}}{4N} && \text{for fixed } z^{\text{Dirac}}; \end{aligned}$$

We can determine the complex density by combining the limiting weight function with the limit of the kernel:

$$K^W(z_1; z_2) \gg \int_0^1 ds s^2 e^{i 2 \rho^2 s^2} f(z_1) J_{\rho+1}(s z_1) J_{\rho}(s z_2); \quad (z_1 \neq z_2)g:$$

We show here $\rho^2 = 0.2$ (with $\rho = 0$):



² We see that the complex eigenvalues almost all occupy a strip of **finite width**, parallel to the real axis.

But for the real and imaginary densities, we find (schematically)

$$\frac{1}{2} \rho_1 = \lim_{N \rightarrow \infty} \frac{1}{2} \rho_N = \lim_{N \rightarrow \infty} \int K_N \in F \in \lim_{N \rightarrow \infty} \int K_N \in F:$$

We will return to this issue in the (F)-377(6)]T0TuF((F)34Tf7.170TD[(:)]TJ

Review