Quartic potential

Squared Bessel process

Multiple orthogonal polynomials

- **The polynomial**

$$B_k(x) = \mathbb{E}\left[\prod_{j=1}^{k}(x - x_j(t))\right],$$

is a monic polynomial of degree k, that satisfies

$$\int_0^\infty B_k(x)x^j w_1(x)\,dx = 0, \qquad \text{for } j = 0, 1, \ldots, \lceil k/2 \rceil - 1,$$

$$\int_0^\infty B_k(x)x^j w_2(x)\,dx = 0, \qquad \text{for } j = 0, 1, \ldots, \lfloor k/2 \rfloor - 1,$$

- **Two weight functions**

$$w_1(x) = x^{\alpha/2} e^{-\frac{Tx}{t(T-t)}} I_\alpha\left(\frac{\sqrt{ax}}{t}\right)$$

$$w_2(x) = x^{\frac{\alpha+1}{2}} e^{-\frac{Tx}{t(T-t)}} I_{\alpha+1}\left(\frac{\sqrt{ax}}{t}\right)$$

Symbol

- The symbol of the Toeplitz matrix is

$$A(z) = z + b + cz^{-1} + dz^{-2}$$

- The equation $\boxed{A(z) = x}$ has three solutions $z_1(x)$, $z_2(x)$, $z_3(x)$ ordered so that

$$|z_1(x)| \geq |z_2(x)| \geq |z_3(x)|$$

- Define sets

$$\Gamma_1 = \{x \in \mathbb{C} \mid |z_1(x)| = |z_2(x)|\}$$

$$\Gamma_2 = \{x \in \mathbb{C} \mid |z_2(x)| = |z_3(x)|\}$$

- FACT 1: The eigenvalues of T_n accumulate as $n \to \infty$ on Γ_1

Schmidt-Spitzer (1960)

Fact 2

- FACT 1: The eighi

Fact 3

- Define μ_2 on Γ_2 by

$$d\mu_2(x) = \frac{1}{2\pi i} \left(\frac{z'_{2-}(x)}{z_{2-}(x)} - \frac{z'_{2}(x)}{z_{2}(x)} \right) dx$$

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- Then μ_2 is a positive measure on2

Family of Toeplitz matrices

- Consider for each $s \geq 0$ the Toeplitz matrix with symbol

$$A_s(z) = z + b(s) + c(s)z^{-1} + d(s)z^{-2}$$

$$= \frac{(z + a(1-t)^2)(z + st(1-t))^2}{z^2}$$

- Then we have solutions of $A_s(z) = x$

$z_1(x, s$ xzs

THEOREM 2

- The fact that $\Gamma_1(s)$ increases leads to the external field V acting on ν_1 with

$$V(x) = \int_0 \log\left(\frac{z_1(x,s)}{z_2(x,s)}\right) ds$$

- The fact that $\Gamma_2(s)$ decreases leads to the upper constraint σ for ν_2 with

$$\sigma = \int_0 \mu_2^s \, ds = \frac{1}{2\pi i} \int_0 \left(\frac{z_{2-}'(x,s)}{z_{2-}(x,s)} - \frac{z_2'(x,s)}{z_2(x,s)} \right) ds$$

- (ν_1, ν_2) is minimizer of the energy functional

$$I(\nu_1) + I(\nu_2) - I(\nu_1,\nu_2) + \int V(x)\, d\nu_1(x)$$

among ν_1 on $[0,\infty)$, $\int d\nu_1 = 1$, ν_2 on $(-\infty, 0]$, $\int d\nu_2 = 1/2$ and

$$\nu_2 \leq \sigma.$$