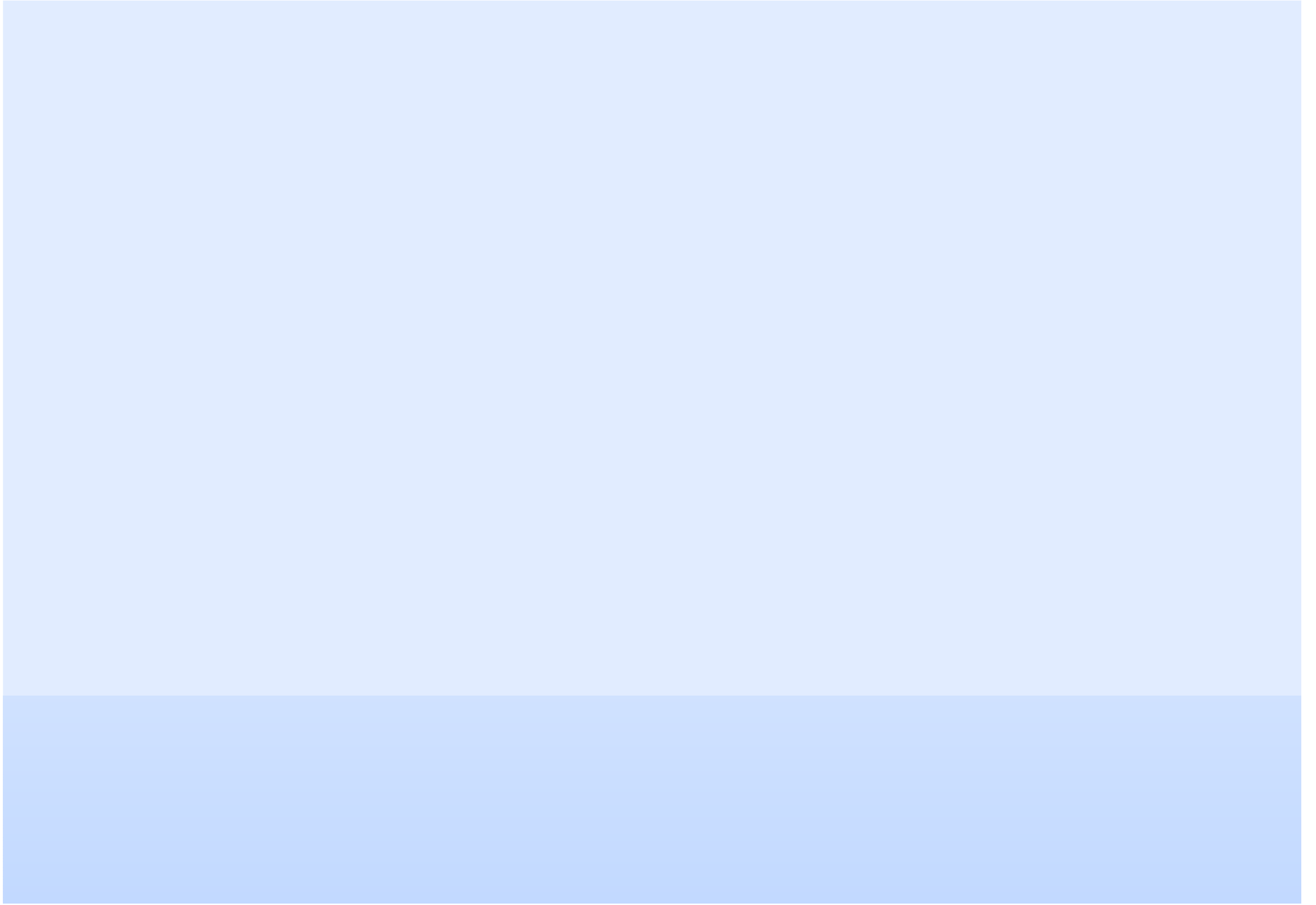
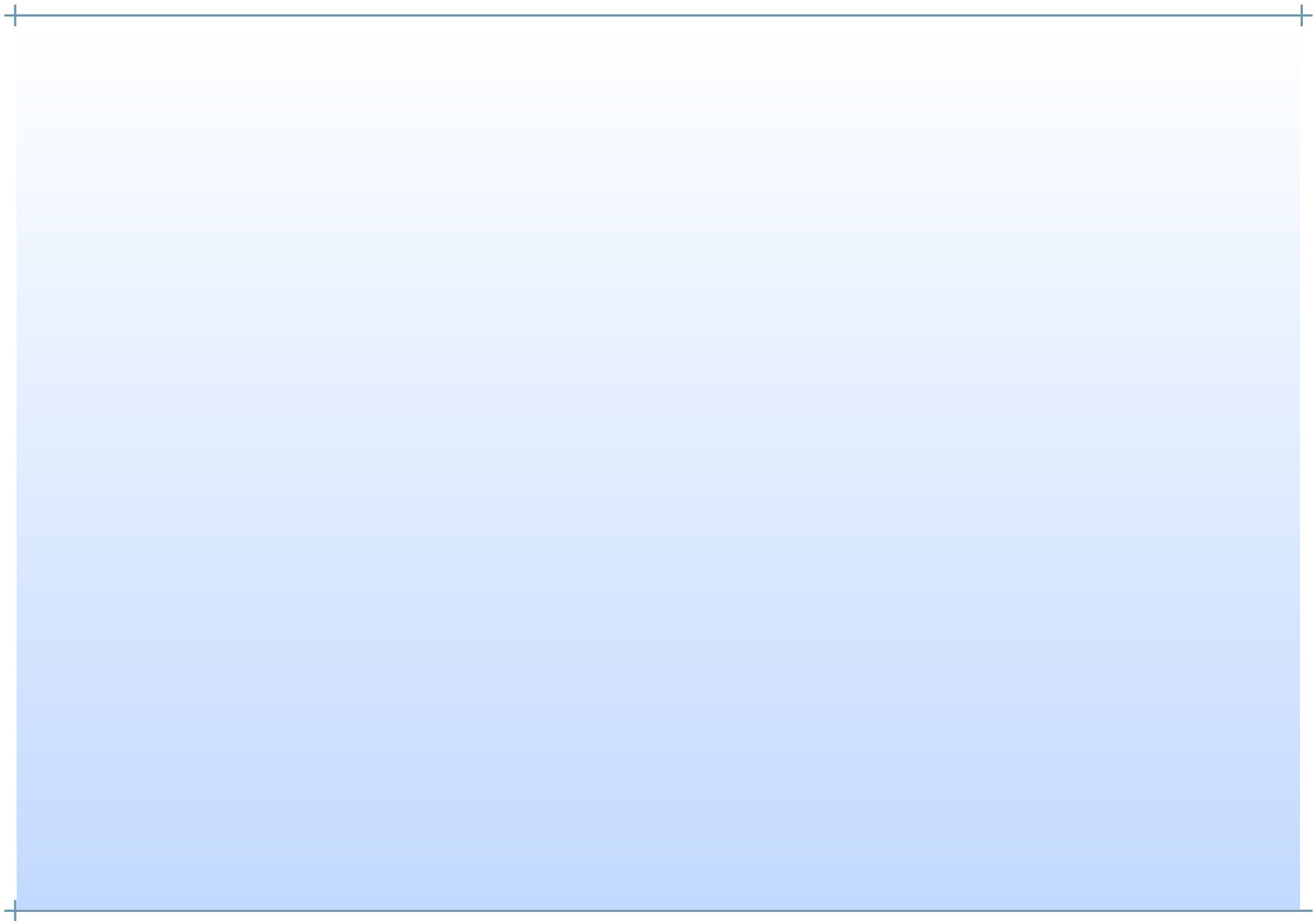
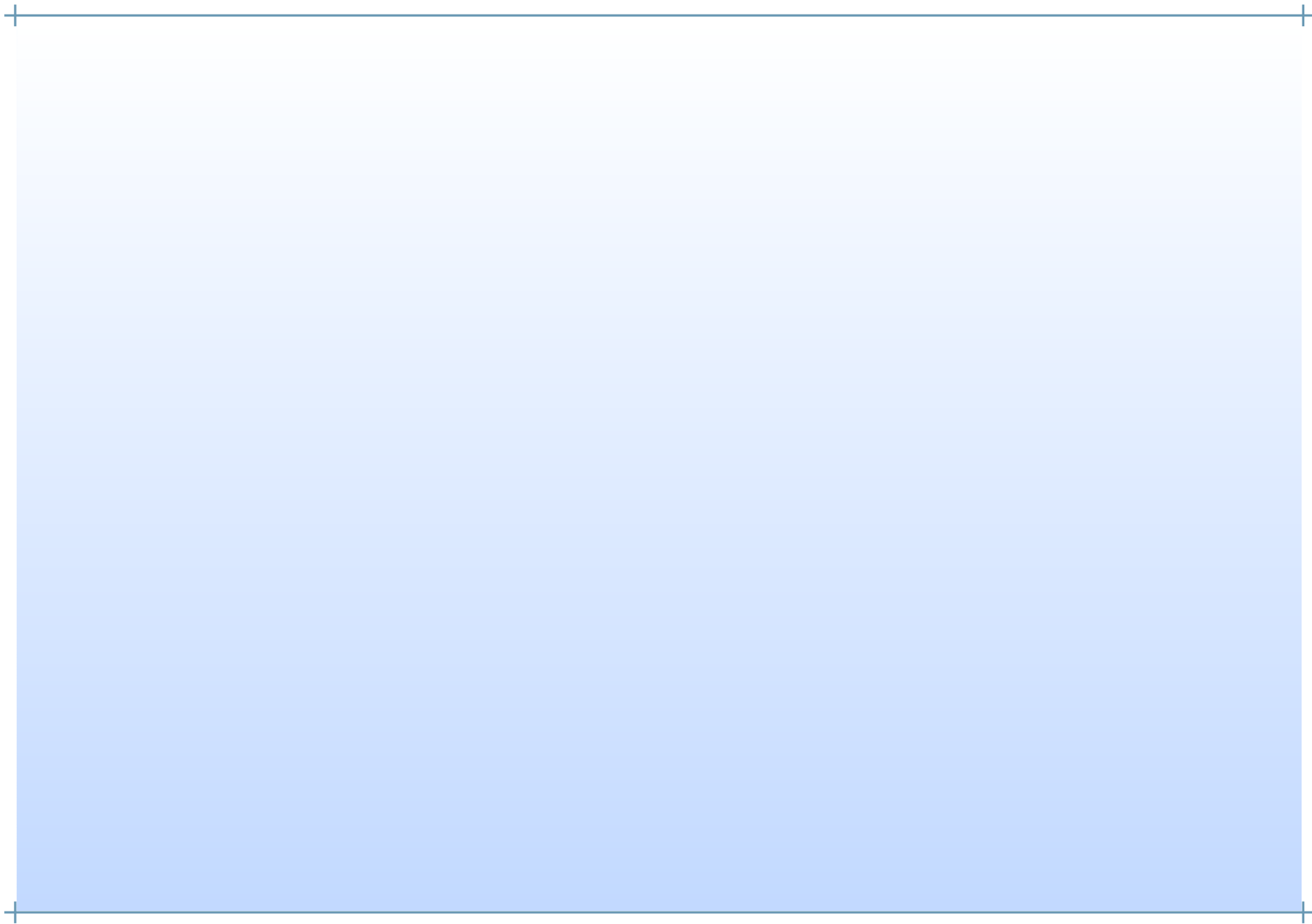


Quartic potential









Squared Bessel process







Multiple orthogonal polynomials

▲ The polynomial

$$B_k(x) = \prod_{j=1}^k (x - x_j(t)) ,$$

is a monic polynomial of degree k , that satisfies

$$\int_0^1 B_k(x) x^j w_1(x) dx = 0, \quad \text{for } j = 0, 1, \dots, \lfloor k/2 \rfloor - 1,$$

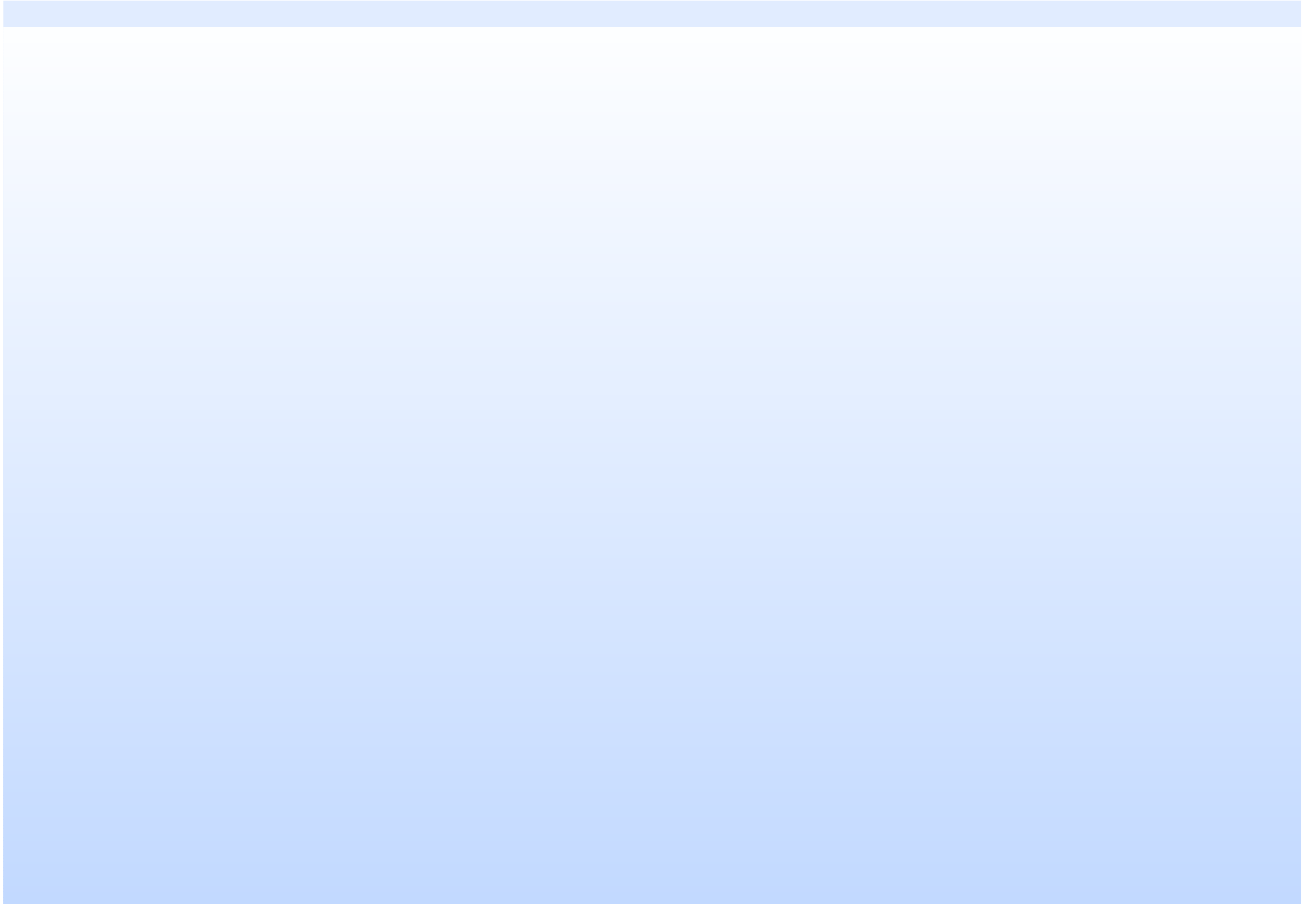
$$\int_0^1 B_k(x) x^j w_2(x) dx = 0, \quad \text{for } j = 0, 1, \dots, \lfloor k/2 \rfloor - 1,$$

▲ Two weight functions

$$w_1(x) = x^{-\alpha} e^{-\frac{\tau x}{t - \tau x}} I_{\alpha} \frac{\sqrt{ax}}{t}$$

$$w_2(x) = x^{-\beta} e^{-\frac{\tau x}{t - \tau x}} I_{\alpha - 1} \frac{\sqrt{ax}}{t}$$







Symbol

- ▲ The symbol of the Toeplitz matrix is

$$A(z) = z + b + cz^{-1} + dz^{-2}$$

- ▲ The equation $A(z) = x$ has three solutions $z_1(x), z_2(x), z_3(x)$ ordered so that

$$|z_1(x)| \geq |z_2(x)| \geq |z_3(x)|$$

- ▲ Define sets

$$\mathcal{I}_1 = \{x \in \mathbb{C} \mid |z_1(x)| = |z_2(x)|\}$$

$$\mathcal{I}_2 = \{x \in \mathbb{C} \mid |z_2(x)| = |z_3(x)|\}$$

- ▲ **FACT 1:** The eigenvalues of T_n accumulate as $n \rightarrow \infty$ on \mathcal{I}_1

Schmidt-Spitzer (1960)

Fact 2

▲ **FACT 1: The eighi**

Fact 3

▲ Define μ_2 on Σ_2 by

$$d\mu_2(x) = \frac{1}{2\pi i} \left(\frac{z'_2(x)}{z_2(x)} - \frac{z'_2(x)}{z_2(x)} \right) dx$$

▲ Then μ_2 is a positive measure on Σ_2

Family of Toeplitz matrices

- ▲ Consider for each $s \geq 0$ the Toeplitz matrix with symbol

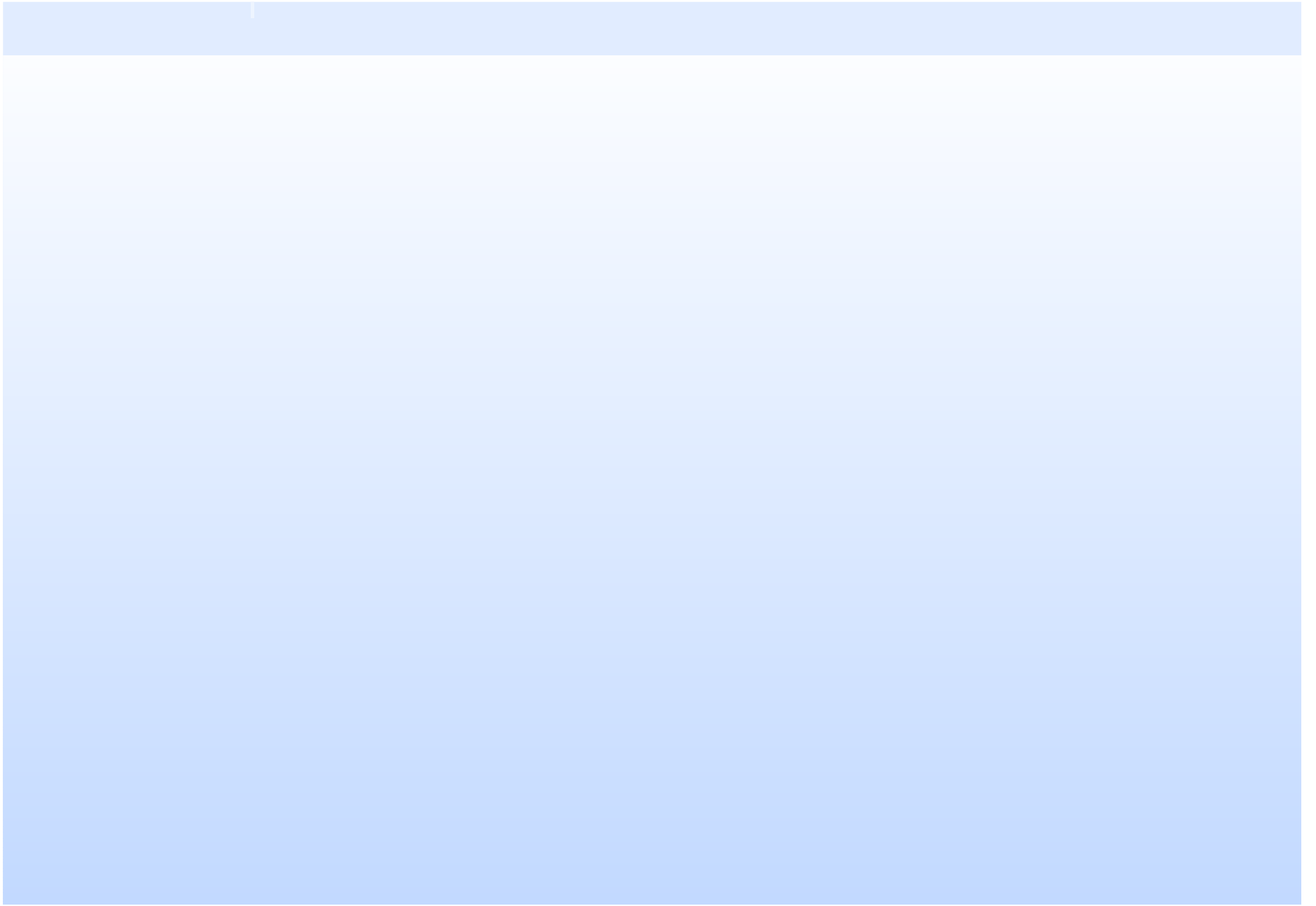
$$\begin{aligned} A_s(z) &= z + b(s) + c(s)z^{-1} + d(s)z^{-2} \\ &= \frac{(z + a(1-t)^2)(z + st(1-t))^2}{z^2} \end{aligned}$$

- ▲ Then we have solutions of $A_s(z) = x$

$$z_1(x, s)$$







THEOREM 2

- ▲ The fact that $\nu_1(s)$ increases leads to the external field V acting on ν_1 with

$$V(x) = \int_0^{\infty} \log \frac{z_1(x, s)}{z_2(x, s)} ds$$

- ▲ The fact that $\nu_2(s)$ decreases leads to the upper constraint σ for ν_2 with

$$\sigma = \int_0^{\infty} \mu_2^s ds = \frac{1}{2\pi i} \int_0^{\infty} \frac{z'_{2-}(x, s)}{z_{2-}(x, s)} - \frac{z'_2(x, s)}{z_2(x, s)} ds$$

- ▲ (ν_1, ν_2) is minimizer of the energy functional

$$I(\nu_1) + I(\nu_2) - I(\nu_1, \nu_2) + \int V(x) d\nu_1(x)$$

among ν_1 on $[0, \infty)$, $d\nu_1 = 1$, ν_2 on $(-\infty, 0]$, $d\nu_2 = 1/2$ and

$$\nu_2 \leq \sigma.$$

