Truncations of random unitary matrices revisited

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joint review with H.-J. Sommers:
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Setup

Choose a unitary matrix at random and partition it:

$$\mathrm{U} = \begin{pmatrix} \mathrm{T} & \mathrm{S} \\ \mathrm{Q} & \mathrm{R} \end{pmatrix} \rightsquigarrow \mathrm{T} \qquad \mathrm{T} \text{ is } \mathrm{m} \times$$

What truncations are good for?

(1) **Quantum transport problems** (Beenakker'97, poster by Nick Simm)

Additive stats of EVs of $\mathrm{T}\mathrm{T}^\dagger$ describe phys quantities of interest, i.e. $\mathrm{tr}\,\mathrm{T}\mathrm{T}^\dagger$ for conductance of quasi one-dimensional wires

(2) **Open chaotic sys** (Fyodorov & Sommers, '97 Życzkowski & S. '00)

Eigenvalues of T are used to model resonances

(3) **Combinatorics of vicious walkers**(Novak '09)

$< |\,\mathrm{tr}\,\mathrm{T}|^N >_T$ enumerates configs of random-turn vicious walkers

(4) **Random determinants** (Fyodorov & K., '07), e.g.,

$$\left\langle \frac{1}{|\det(\mathbf{I} - \mathbf{z}\mathbf{A})|^2} \right\rangle_A = \int \left\langle \frac{1}{\det(\mathbf{I} - |\mathbf{z}|^2 \mathrm{T}\mathrm{T}^\dagger \quad \mathbf{A}\mathbf{A}^\dagger)} \right\rangle_A d\rho_n \quad \times \quad (\mathrm{T})$$

for complex random $\mathrm{n} \times \mathrm{n}$ matrices A with invariant distribution.

Singular values of T (1,4); eigenvalues of T (2,3)

Matrix measure

Truncation map: $U \mapsto T$, U is $n \times n$, T is $m \times p$, $m \le p$

Have $TT^\dagger + SS^\dagger = I$ by unitarity. If $n \geq m + p$ then (generically) $SS^\dagger$ has rank m and the image of $U(n)$ is the entire matrix ball $TT^\dagger \le I$.

Theorem 1 *(Friedman&Mello '85, Fyodorov&Sommers '03, Forrester '06)*

For $n \geq m + p$

$$d\rho_{n, m\times p}(T) \propto \det(I - TT^\dagger)^{n-m-p}\chi_{TT^\dagger \le I}(T)dT$$

where dT *is the Cartesian volume element in* $\mathbf{C}^{m\times p}$*. For invariant* f

$$\int_{\mathbf{C}^{m\times p}} f(TT^\dagger)d\rho_{n, m\times p}(T) = const. \times$$

$$= \int_{\mathbf{C}^{m\times m}} f(ZZ^\dagger)\det(ZZ^\dagger)^{p-m}\det(I - ZZ^\dagger)^{n-m-p}\chi_{ZZ^\dagger \le I}(Z)dZ$$

Matrix measure

Eigenvalue correlation functions

These are just marginals of the jpdf:

$$R\ (z_1,\dots,z\ ) = \frac{m!}{(m-k)!}\int d^2z_{\ +1}\dots\int d^2z\ \ P(z_1,\dots,z\ ,z_{\ +1},\dots,z\ \ ),$$

The EV corr fncs for truncations can be obtained by the method of OPs.

For the rotation invariant weights, $w(z) = w(|z|)$, OPs are just powers z^l :
$\int d^2z\, w(z)\, z\ z^{*}\ \ = h\ \delta$, leading to

$$R\ (z_1,\dots,z\ ) = \prod_{l=1} w(z_l\ )\ \det(K(z\ ,z\ )); \qquad K(u,v) = \sum_{l=0}^{-1}\frac{(uv^*)^l}{h_l}$$

For truncated unitaries the sum on the rhs is the **binomial series** for $(1-uv^*)^{-(n-\ \ +1)}$ truncated after m terms. This gives the kernel in terms of the incomplete Beta function.

Kernel

Incomplete Beta fnc:

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x t^{\ -1}(1-t)^{\ -1}\, dt$$

We have

$$K(u,v) = \frac{n-m}{\pi} \frac{I_{1-}\quad(n-m+1,m)}{(1-uv^*)^{n- \quad +1}}$$

This representation seems to be new. It is convenient for asymptotic analysis. Also one can handle more general $w(z) = |z|^{2}(1-|z|^2)$.

Compare with the complex Ginibre ($d\mu(J)\quad e^{-\,tr}\quad dJ$). There $w(z) = e^{-|\ |^2}$, have truncated exponential series for the kernel:

$$K(u,v) = \frac{n}{\pi} e \quad \frac{\Gamma(n,uv^*)}{\Gamma(n)} \quad \text{with} \quad \Gamma(n,x) = \int_x^\infty e^{-}\ t^{n-1} dt$$

Strong non-unitarity - EV density in the bulk

Strong non-unitarity: boundary of EV distribution

Strong non-unitarity, locally at the origin

Scale z

Weak non-unitarity: EV density and correlations

Scaling z accordingly, $z = \left(1 - \frac{y}{m}\right) e^{i\varphi_0 + i\frac{\varphi_j}{m}}$, one finds the EV density

$$R_1(z) \quad \frac{m^2}{\pi} \frac{(2y)^{l-1}}{(l-1)!} \int_0^1 e^{-2 \quad t} t^{l} \, dt \, , \quad m \to \quad \text{and } l \text{ is finite.}$$

(Życzkowski & Sommers, ’00) and correlations

$$R \ (z_1, \ldots, z \ ) \quad \left(\frac{m^2}{\pi}\right) \prod_{=1} \frac{(2y \ )^{l-1}}{(l-1)!} \det\left(\int_0^1 e^{-(\ _i + \ _j + i(\varphi_i - \varphi_j))} t^{l} \, dt\right)$$

This is a particular case of a ’universal’ expression describing EV correlations for random contractions (Fyodorov & Sommers ’03).

Interestingly, a different ensemble, $J = H + i\gamma W$, leads to the same form of correlations (Fyodorov & K. ’99). Here H is drawn from the GUE, $\gamma > 0$ and W is a diagonal matrix with l 1’s and m zeros.

Conclusion