Applications of Free Random Variables to Financial Analysis

Andrzej Jarosz andrzej.jarosz@clico.pl Clico Ltd., Kraków, Poland

in collaboration with Z. Burda, J. Jurkiewicz, M. A. Nowak, M. Snarska (Jagiellonian University, Kraków, Poland), G. Papp (Eötvös University, Budapest, Hungary), I. Zahed (SUNY Stony Brook, USA)

Abstract

We apply the concept of *free random variables* (FRV), which is a noncommutative extension of probability calculus, to *doubly–correlated Gaussian Wishart* random matrix models, appearing for example in multivariate analysis of financial time series displaying both inter–asset cross–covariances $\mathbf{C}$ (such as in factor models) and temporal auto–covariances $\mathbf{A}$ (such as in models with heteroscedasticity, in VARMA processes, *etc.*).

Correlated Gaussians

We consider a universe of N financial assets ($i = 1, \ldots, N - 1$), sampled over T time moments ($a = 1, \ldots, T - 1$): R_{ia} is *e.g.* the demeaned logarithmic return. The simplest approximations: $\mathbf{R}$ is a *Gaussian* random matrix with the structure of covariances,

$$\langle R_{ia} R_{jb} \rangle = C_{ij} A_{ab}. \tag{1}$$

The change of variables $\mathbf{R} = \sqrt{\mathbf{C}}\,\tilde{\mathbf{R}}\sqrt{\mathbf{A}}$ gives uncorrelated Gaussians $\tilde{\mathbf{R}}$.

Problem: *estimation* of $\mathbf{C}$ from historical time series. Marred by the measurement *noise*, quantified by $r \equiv N/T$. The Pearson estimator of $\mathbf{C}$,

$$\mathbf{c} = \frac{1}{T}\sqrt{\mathbf{C}}\,\tilde{\mathbf{R}}\mathbf{A}\tilde{\mathbf{R}}^{T}\sqrt{\mathbf{C}}. \tag{2}$$

Free Random Variables

Voiculescu and Speicher's [4] *free random variables calculus* is a generalization of probability theory to *noncommutative random variables*, such as infinite (Hermitian) random matrices $\mathbf{X}$. It relies on the concept of *freeness*, which is noncommutative independence.

Classical probability:	**Noncommutative probability (FRV):**
p.d.f., $P_X(x)$	spectral density, $\rho_{\mathbf{X}}(\lambda)$
characteristic function, $g_X(z) \equiv \langle e^{izX} \rangle$	Green's function, $G_{\mathbf{X}}(z) \equiv (1/N)\langle \mathrm{Tr}\, 1/(z\mathbf{1}_N - \mathbf{X}) \rangle$, or M–transform, $M_{\mathbf{X}}(z) = zG_{\mathbf{X}}(z) - 1$
independence	freeness
Addition of independent commutative random variables: The logarithm of the characteristic function, $r_X(z) \equiv \log g_X(z)$, is additive, $$r_{X_1+X_2}(z) = r_{X_1}(z) + r_{X_2}(z). \quad (3)$$	**Addition of free noncommutative random variables:** The Blue's function, $G_{\mathbf{X}}(B_{\mathbf{X}}(z)) = B_{\mathbf{X}}(G_{\mathbf{X}}(z)) = z$, is additive, $$B_{\mathbf{X}_1+\mathbf{X}_2}(z) = B_{\mathbf{X}_1}(z) + B_{\mathbf{X}_2}(z) - \frac{1}{z}. \quad (4)$$
Multiplication of independent r.v.: Reduced to the addition problem via the exponential map, owing to $e^{X_1}e^{X_2} = e^{X_1+X_2}$.	**Multiplication of free r.v.:** The N–transform, $M_{\mathbf{X}}(N_{\mathbf{X}}(z)) = N_{\mathbf{X}}(M_{\mathbf{X}}(z)) = z$, is multiplicative,