Physic pp ic tions of the NLSE

- **Shallow water waves**
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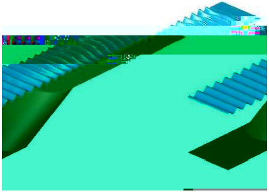
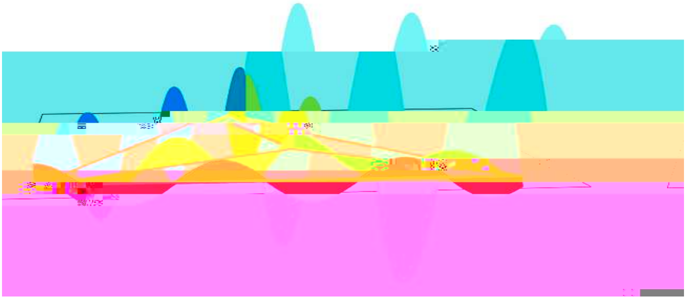


es on etric gr phs



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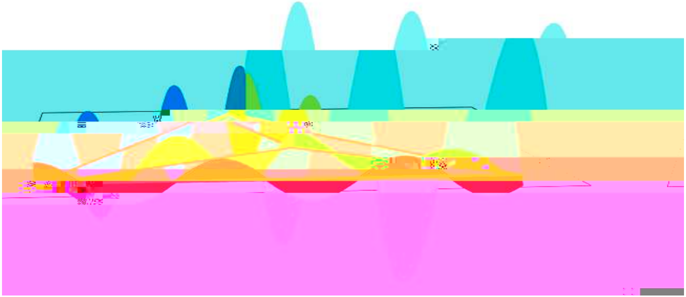

- **linear waves: quantum graphs**
  Schrödinger/Helmholtz equation, Dirac equation, Bogoliubov-de Gennes equation.
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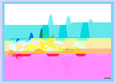
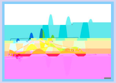


he st tion ry NLSE in one di ension

$$-\psi''(x) + g|\psi(x)|\;\psi(x) = E\psi(x)$$



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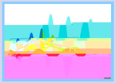
- coupling constant g
  may be positive (repulsive) or negative (attractive)
- 'energy' E
- integrable, stationary solutions are known (Carr *et al*

C ssic dyn ics pict re for NLSE for $E > \mathbf{0}$

$$\tau = \sqrt{E}x, \qquad \psi(x) = \sqrt{\frac{2E}{|g|}}\, r(\tau) e^{i\theta(\tau)}, \qquad \sigma = \frac{g}{|g|}$$

NLSE is generated by Lagrangian

$$L = \frac{\dot{r} \quad + r \quad \dot{\theta}}{2} - \frac{r \quad + \sigma r}{2}$$



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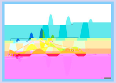
conjugate momenta

- $p = \frac{\partial L}{\partial \dot{r}} = \dot{r}$: radial momentum
- $\ell = \frac{\partial L}{\partial \dot{\theta}} = r\ \dot{\theta}$: angular momentum (conserved)
- angular momentum $\longrightarrow$ flux
  $\psi^* \psi' = \frac{E\ /\ }{|g|}(rp + i\ell)$
- If $\ell = 0$ one may choose ψ to be real



Two constants of motion

$$\ell = r \; \dot{\theta} \qquad \mathcal{E} \; = \; \dot{r}$$

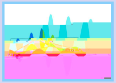
Two constants of motion

$$\ell = r \ \dot{\theta} \qquad \mathcal{E}_{V} \ldots = \frac{\dot{r}}{2} + V_{\text{eff}}(r, \ell)$$

with e□ective potential

$$V_{\text{eff}}(r; \ell) = \frac{\ell}{\ }$$

NLSE on etric gr ph



- vertices $i = 1, 2, \ldots, V$
- edges may either be finite **bonds** $b \equiv (i, j)$ that connect two vertices i and j
  or infinite **leads** $l = (i, \infty)$ that start a vertex i and go to infinity.

NLSE on etric gr ph

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M tching conditions

- Consider one vertex with v incident edges and coordinates $x_1, x\ , \ldots, x_{\mathbf{v}}$ such that $x_{\mathbf{i}} = 0$ at the vertex and $g_{\mathbf{i}} = g_{\mathbf{j}} = g$.
- **Contin ity**:

$$\psi_{\mathbf{i}}(0) = \psi_{\mathbf{j}}(0) = \psi_0$$

M tching conditions

- Consider one vertex with v incident edges and coordinates $x_1, x\ , \ldots, x_{\mathbf{v}}$ such that $x_{\mathbf{i}} = 0$ at the vertex and $g_{\mathbf{i}} = g_{\mathbf{j}} = g$

F conser tion

F conser tion

Robin-type matching conditions in terms of the classical dynamics picture (with $\psi_0 = \sqrt{2E/|g|}\, r_0 e^{\mathbf{i}\theta_0}$)

$$\sqrt{E}\sum_{\mathbf{j}=1}^{\mathbf{v}}\left(\dot{r}_{\mathbf{j}} + i r_{\mathbf{j}}\dot{\theta}_{\mathbf{j}}\right) e^{\mathbf{ti}\theta_{\mathbf{j}}} = \lambda r_0 e^{\mathbf{i}\theta_0}$$

where the left-hand side is evaluated at the vertex $x_{\mathbf{j}} = 0$ ($\tau_{\mathbf{j}} = 0$).

$$\sum_{\mathbf{j}=1}^{\mathbf{v}} p_{\mathbf{j}} = \frac{\lambda}{\sqrt{E}}$$

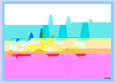
$$\sum_{\mathbf{j}=1}^{\mathbf{v}} \ell_{\mathbf{j}} = 0$$

Sum of fluxes (outgoing) at each vertex vanishes.

Ho to nd the so tion

Consider a finite graph with V vertices and B bonds (no leads) and fix an energy $E > 0$.

Ho to nd the so tion

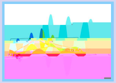


Ho to nd the so tion

Consider a finite graph with V vertices and B bonds (no leads) and fix an energy $E > 0$.
The wave function $\psi_b(x_b)$ on bond b is characterised by two parameters (e.g. ℓ_b, $\mathcal{E}_{\mathrm{Ham},b}$)

1. Given a set of V complex numbers ϕ_i = find ℓ_b and $\mathcal{E}\mathrm{Ham}, b$ such that $\psi_b(x_b = 0) = \phi_i$ and $\psi_b(L_b) = \phi_j$.
   Classical picture: find trajectories that connect to two points in configuration space in a given time.
- The Robin-type matching conditions are then a set of v equations for v unknowns ϕ_i.
  Each solution $\{\phi_i\}$ yields a wave function Ψ on the nonlinear graph.

Sc ttering fro n non ine r gr ph



- The physical relevance of the ’spectrum’ is questionable apart from the ground state energy
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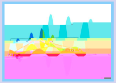

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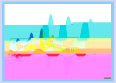

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