Critical asymptotics for Toeplitz determinants

Tom Claeys

Université de Lille 1

Brunel RMT workshop

December 2009

Joint work with A. Its and I. Krasovsky

Toeplitz determinants

- Toeplitz matrix = matrix which is constant along diagonals

$$\begin{pmatrix} c_0 & c_{-1} & c_{-2} & \cdots & c_{-n+1} \\ c_1 & c_0 & c_{-1} & \ddots & \vdots \\ c_2 & c_1 & \ddots & \ddots & c_{-2} \\ \vdots & \ddots & \ddots & c_0 & c_{-1} \\ c_{n-1} & \cdots & c_2 & c_1 & c_0 \end{pmatrix}$$

- Toeplitz determinant is the determinant of a Toeplitz matrix
- Asymptotics for Toeplitz determinants when the size of the matrices tends to infinity?

Toeplitz determinants

-

Toeplitz determinants

- If the weight f θ
  - is "smooth"
  - has no zeros
  - has a continuous logarithm (winding number around the origin)
- Szegő's strong limit theorem: as n ,

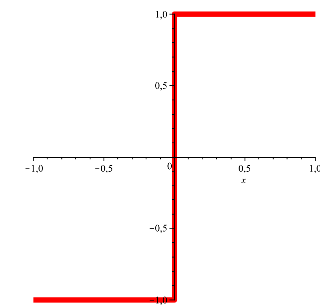
$$\ln D_n\ f \qquad n \ln f\ {}_0 + \sum_{k=1} k \ln f\ {}_k \ln f\ {}_k + o \qquad ,$$

with

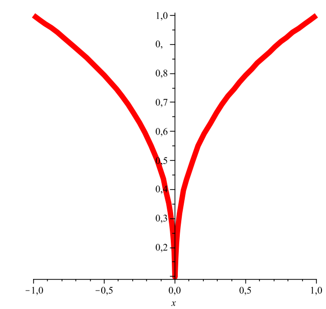
$$\ln f\ {}_k \qquad \frac{\ }{\pi}\ {}_0^{2\pi} \ln f\ e^{i\theta}\ e^{\ ik\theta} d\theta.$$

Fisher-Hartwig singularities

- Two types of weights for which Szegő asymptotics are not valid
  - ► jump discontinuities



  - ► root type singularities



- Example

$$f \quad e^{i\theta} \qquad \cos\theta \;\; {}^{\alpha} e^{i\beta(\theta \quad \pi)} e^{V(e^{i\theta})}, \qquad \text{for} \quad < \theta < \quad \pi,$$

with $\mathrm{Re}\,\alpha > \quad 1$

Fisher-Hartwig singularities

- For weights with one Fisher-Hartwig singularity with parameters α (root) and β (jump),

$$\ln D_n \; f \qquad nV_0 + \sum_{k=1}^{\infty} kV_kV_{-k} \quad \alpha \quad \beta \sum_{k=1}^{\infty} V_k \quad \alpha + \beta \sum_{k=1}^{\infty} V_{-k}$$

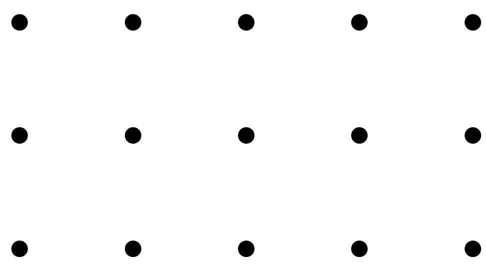
$$+ \;\; \alpha^2 \quad \beta^2 \; \ln n + \ln \frac{G \quad + \alpha + \beta \; G \quad + \alpha \quad \beta}{G \quad + \;\; \alpha} + o \qquad ,$$

as n , where G is Barnes' G-function, and

$$V_k \qquad —$$

2d Ising model

- lattice with an associated spin variable taking values $\pm$ at each point of the lattice



2d Ising model

- 2-spin correlation functions are Toeplitz determinants:

$$< \sigma_{00}\sigma_{0k} > \quad D_k \; f \; ,$$

for a certain symbol f

  - For $T < T_c$

Transition from Szegő to FH

Transition from Szegő to FH

- Asymptotics as n

Transition from Szegő to FH

$$\mathsf{nt} \quad \int_0^{2\mathsf{nt}} \left(\mathsf{w} \;\; \mathsf{x} \qquad \frac{\alpha^2 \qquad \beta^2}{\mathsf{x}} \right) \mathsf{dx} + \;\; \alpha^2 \qquad \beta^2 \; \ln \;\; \mathsf{nt},$$

$$\mathsf{w} \;\; \mathsf{x} \qquad \frac{}{\mathsf{x}} \quad \int_{\mathsf{x}}^{+\infty} \mathsf{v} \;\; \xi \;\; \mathsf{d}\xi,$$

- v

Asymptotics

$$v\ x \quad \begin{cases} \mathcal{O} \qquad + \mathcal{O}\ x^{2\alpha} \ + \mathcal{O}\ x^{2\alpha}\ln x\ , & x \qquad , \\ \mathcal{O}\ e^{-cx}\ , & x \qquad + \qquad , \end{cases}$$

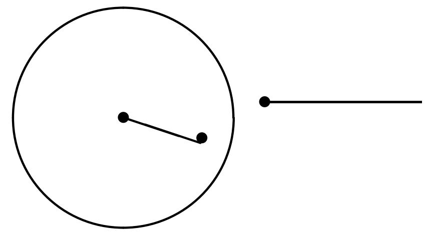
$$_0^{+} \quad v\ x\ dx \qquad \alpha^2 \qquad \beta^2.$$

$$w\ x \quad \begin{cases} \frac{\alpha^2-\beta^2}{x} + \mathcal{O} \qquad + \mathcal{O}\ x^{2\alpha} \ + \mathcal{O}\ x^{2\alpha}\ln x\ , & x \qquad , \\ \mathcal{O}\ e^{-cx}\ , & x \qquad + \qquad . \end{cases}$$

$$x \quad \begin{cases} \alpha^2 \qquad \beta^2 \ \ln x + \mathcal{O}\ x\ , & x \qquad , \\ \quad \ln \frac{G(1+\alpha+\beta)G(1+\alpha-\beta)}{G(1+2\alpha)} + \mathcal{O}\ e^{-cx}\ , & x \qquad + \qquad . \end{cases}$$

Transition from Szegő to FH

- Extension to complex t?



Expansion is valid for $\arg t < \frac{\pi}{2}$ if contour of integration does not contain poles of w

- different choices of contour different branches of logarithm

Transition from Szegő to FH

- what if Im α and/or Re β ?
  - w x α, β is not real for x >
  - w can have poles on , +
  - asymptotic expansion holds only if we integrate over a pole-free contour
    expansion not valid if nt is a pole of w x α, β
  - poles correspond to Toeplitz determinants approaching
    different choices of integration contour
    expansion picks up residue of w
    residue of w 4.797(p)-0.5a3.959z246]TJ /R28 20

Orthogonal polynomials

- Heine's formula: determinant formula for orthogonal polynomials

p_{nab}

Asymptotics for Toeplitz determinants

General approach to obtain asymptotics for Toeplitz determinants for weight f

- Step 1: deform weight f smoothly to a weight for which Toeplitz determinant is known (e.g. uniform weight),

$$f_t \; z \;, \qquad f_1 \; z \qquad f, \qquad f_0 \; z$$

- Step 2: try to find differential identity for $\frac{d}{dt}$

Transition from Szegő to FH

Applied to our transition between Szegő and FH

- Step 1: deformation of weight:

$$f_{t} \;\; z \qquad z \qquad e^{t \;\; \alpha+\beta} \;\; z \qquad e^{\;\; t \;\; \alpha}$$

- Step 2: differential identity

$$\frac{d}{dt}\ln D_n \; t \qquad \alpha+\beta \; e^{t}\left(Y^{-1}Y'\right)_{22} \; e^{t} + \alpha \quad \beta \; e^{-t}\left(Y^{-1}Y'\right)_{22} \; e^{-t}$$

where

$$Y \; z \qquad \begin{pmatrix} \chi_n^{-1} p_n \; z & p_n^{-1} \; {}_{C_1} \frac{p_n(\xi)}{\xi - z} \frac{f(\xi) d\xi}{2\pi i \xi^n} \\ \chi_{n-1} z^{n-1} p_{n-1} \; z^{-1} & \chi_{n-1} \; {}_{C_1} \frac{p_{n-1}(\xi^{-1})}{\xi - z} \frac{f(\xi) d\xi}{2\pi i \xi} \end{pmatrix}$$

- Y is solution of the Riemann-Hilbert problem for orthogonal polynomials